Solution to Assignment 8

Supplementary Exercises

1. Use the Weierstrass M-test to study the uniform convergence of the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ for $x \in (0,b)$ where $b > 0$. The answer depends on the value of b.

   Solution. This series converges uniformly on any interval of the form $[0,b], b \in (0,1)$ as a direct application of the M-Test. It is not convergent at $x = 1$, hence it cannot be uniformly convergent on $[0,b]$ when $b \geq 1$. We will see that this series has a closed form given by $-\log(1-x)$.

2. Show that the series $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^p}$ defines a continuous function on $\mathbb{R}$ for $p > 1$.

   Solution. By
$$\left|\frac{\sin nx}{n^p}\right| \leq \frac{1}{n^p}$$
   and $\sum n^{-p}$ is convergent if $p > 1$ we conclude from M-Test that this series is uniformly convergent on $\mathbb{R}$. As uniform convergence preserves continuity, the limit $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^p}$ is a continuous function.

3. Show that the infinite series $\sum_{j=1}^{\infty} \dfrac{\cos 2^j x}{3^j}$ is a continuous function on the real line. Is it differentiable?

   Solution. By
$$\left|\frac{\cos 2^j x}{3^j}\right| \leq \frac{1}{3^j}$$
   and $\sum_j 3^{-j} < \infty$, we conclude from M-Test that this series is uniformly convergent. By Continuity Theorem 3.6' we further deduce that it is continuous. In fact, this function has continuous derivative given by
$$-\sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^j \sin 2^j x \ .$$
   To see this just observe that $\sum_j (2/3)^j < \infty$ and then appeal to Differentiation Theorem 3.8'.

4. Show that the sequence $g_n(x) = \sum_{j=1}^{n} e^{-jx}$ defines a smooth function on $[1,,\infty)$. What will happen if $[1,\infty)$ is replaced by $[0,\infty)$?

   Solution. By termwise differentiating this series k times we get the series
$$\sum_{j=1}^{\infty} (-1)^k j^k e^{-jx} \ .$$
   To show that the series defines a smooth function on $[1,\infty)$ it suffices to show these derived series are uniformly convergent and then apply Differentiation Theorem repeatedly. We recall the inequality
$$e^x \geq \frac{x^n}{n!}$$

for all positive x and n. In particular, taking x to be jx we get

$$e^{-jx} \leq n!(jx)^{-n} \leq n!j^{-n} ,$$

for $x \in [1, \infty)$. We choose $n = k + 2$ to get $j^k e^{-jx} \leq (k+2)!j^{-2}$. Since $\sum_j j^{-2} < \infty$, this series converges uniformly on $[1, \infty)$.

When considering the series on $[0, \infty)$, we note that it is not convergent at $x = 0$, so not even pointwise convergence, let alone uniform convergence.

Note: By slightly modifying the proof, this series is uniformly convergent on $[b, \infty)$ for every $b > 0$ and hence smooth on $(0, \infty)$.

5. Suppose f is a nonzero function satisfying $f(x+y) = f(x)f(y)$ for all real numbers x and y and is differentiable at $x = 0$. Show that it must be of the form e^{ax} for some number a. Hint: Study the differential equation f satisfies.

   Solution. Taking $x = y = 0$ in the defining relation of f we get $f(0) = f(0)^2$, so $f(0) = 0$ or $f(0) = 1$. We first exclude the case $f(0) = 0$. Indeed, from the relation $f(x+y) = f(x)f(y)$ one deduces $f(x) = f(x/n)^n$. If $f(0) = 0$, as $f'(0)$ exists, one has $\lim_{h \to 0} f(h)/h = 0$. For $\varepsilon = 1$, there exists some δ $|f(h)| \leq |h|$ for all $|h| < \delta$. For any x we can find a large n_0 such that $|x|/n_0 < \delta$. Then $|f(x)| = |f(x/n)|^n \leq (x/n)^n$ for all $n \geq n_0$. Letting $n \to \infty$, we conclude $f(x) = 0$ for all x, contradicting the assumption that f is non-zero. We have shown that $f(0) = 1$. Now,

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)(f(h) - 1)}{h} \to f(x)f'(0) , \quad \text{as } h \to 0 ,$$

   which shows that f is differentiable everywhere with $f'(x) = f'(0)f(x)$. Setting $a = f'(0)$, the function $g(x) = f(x/a)$ satisfies $g'(x) = a^{-1}f'(x/a) = a^{-1}f'(0)f(x/a) = g(x)$ and $g(0) = f(0) = 1$. By the uniqueness of the exponential function we conclude that $g(x) = E(x)$, so $f(x) = E(ax)$.

6. (a) Show that

$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} \leq E(x) \leq 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^a x^n}{n!} , \quad x \in [0, a] .$$

   (b) Show that e is not a rational number. Suggestion: Deduce from (a) the inequality

$$0 < en! - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) n! < \frac{e}{n+1} .$$

   Solution. (a) A direct application of Taylor's Expansion Theorem.

   (b) It follows from (a) by noting $e^a \leq e$ for $a \in (0, 1]$. If $e = p/q$ for some $p, q \in \mathbb{N}$, the inequality gives

$$0 < \frac{p}{q}n! - \left(1 + \frac{1}{2} + \cdots \frac{1}{n!}\right) n! < \frac{e}{n+1} < 1 .$$

   The term

$$\left(1 + \frac{1}{2} + \cdots \frac{1}{n!}\right) n!$$

   is a natural number. When $n = q$ the term $\frac{p}{q}n!$ is also a natural number, so is their difference. But there is no natural number lying between 0 and 1 ! BTW, it is much more difficult to show e is a transcendental number. Google for it.

7. Show that the series
$$\sum_{j=0}^{\infty} \frac{x^j}{j!}$$
is not uniformly convergent on $\mathbb{R}$ (although it is uniformly convergent in every $[-M, M]$).
Solution. Were it uniformly convergent on $(-\infty, \infty)$, for $\varepsilon > 0$, there is some n_0 such that
$$\left| \sum_{j=0}^{n} \frac{x^j}{j!} - \sum_{j=0}^{m} \frac{x^j}{j!} \right| < \varepsilon, \quad \forall n, m \geq n_0 \ , \forall x \in (-\infty, \infty).$$
Taking $n = n_0 + 1$ and $m = n_0$, we have
$$\left| \frac{x^n}{n!} \right| < \varepsilon \ , \quad x \in (-\infty, \infty) \ ,$$
which is absurd. Hence the convergence cannot be uniform.

8. (a) Show that
$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t} dt \ .$$
Suggestion: Think about
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \frac{(-x)^n}{1+x} \ .$$
(b) Show that
$$\left| \log(1 + x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} \right) \right| \leq \frac{x^{n+1}}{n+1} \ .$$

Solution. (a) follows from a direct integration. The second inequality follows from the first inequality after noting
$$\left| \int_0^x \frac{(-t)^n}{1+t} dt \right| \leq \int_0^x t^n dt = \frac{x^{n+1}}{n+1} \ .$$

9. This exercise suggests an alternative way to define the logarithmic and exponential functions. Define $\mathrm{nog} : (0, \infty) \to \mathbb{R}$ by
$$\mathrm{nog}(x) = \int_1^x \frac{1}{t} dt.$$
(a) $\mathrm{nog}(x)$ is strictly increasing, concave, and tends to ∞ and $-\infty$ as $x \to \infty$ and 0 respectively.
(b) $\mathrm{nog}(xy) = \mathrm{nog}(x) + \mathrm{nog}(y)$.
(c) Define $e(x)$ to be the inverse function of nog. Show that it coincides with $E(x)$.

Note: f is concave means $-f$ is convex. You cannot assume $\log x$ has been defined.

Solution.

(a) By fundamental theorem of calculus, nog is differentiable and $(\text{nog } x)' = \frac{1}{x} > 0$. Hence it is strictly increasing. Moreover, $(\text{nog } x)'' = -\frac{1}{x^2} < 0$ hence it is strictly concave. Next we observe $\forall\ x \geq 2$, $\exists\ n_x \in \mathbb{R}$ s.t. $n_x - 1 \leq x < n_x$. Then

$$\begin{aligned} \text{nog } x \geq \text{nog}(n_x - 1) &= \int_1^{n_x-1} \frac{1}{t} \\ &= \sum_{k=2}^{n_x-1} \int_{k-1}^{k} \frac{1}{t} \geq \sum_{k=2}^{n_x-1} \int_{k-1}^{k} \frac{1}{k} \\ &= \sum_{k=2}^{n_x-1} \frac{1}{k} \ . \end{aligned}$$

Letting $x \to \infty$, $n_x \to \infty$, hence $\lim_{x\to\infty} \text{nog } x \geq \sum_{k=2}^{\infty} \frac{1}{k} = \infty$.
Next, by the change of variables $s = 1/t$,

$$\text{nog } x = \int_1^x \frac{dt}{t} = \int_{1/x}^{1} \frac{ds}{s} \to -\infty \ ,$$

as $x \to 0$.

(b)

$$\begin{aligned} \text{nog } xy &= \int_1^{xy} \frac{1}{t}\, dt \\ &= \int_1^{x} \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{t}\, dt \\ &= \int_1^{x} \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{xt}\, d(xt) \ , \end{aligned}$$

since $x > 0$. It is equal to

$$\int_1^{x} \frac{1}{t}\, dt + \int_1^{y} \frac{1}{u}\, du = \text{nog } x + \text{nog } y \ .$$

(c) From (a), nog is strictly increasing hence one-to-one, its inverse function $e(x)$ is well defined.

$$e'(x) = \frac{1}{(\text{nog})'(e(x))} = \frac{1}{1/e(x)} = e(x) \ \ \forall\ x \in \mathbb{R} \ ,$$

and $e(0) = 1$ since nog $(1) = 0$. By uniqueness, $e(x)$ coincides with $E(x)$.
Note. This approach has a drawback, namely, it is not feasible for generalization.

10. Show that there is a unique solution $c(x), x \in \mathbb{R}$, to the problem

$$f'' = f, \quad f(0) = 1, \ f'(0) = 0.$$

(a) Letting $s(x) \equiv c'(x)$, show that s satisfies the same equation as c but now $s(0) = 0, \ s'(0) = 1$.

(b) Establish the identities, for all x,

$$c^2(x) - s^2(x) = 1,$$

and

$$c(x + y) = c(x)c(y) + s(x)s(y).$$

(c) Express c and s as linear combinations of e^x and e^{-x}. (c and s are called the hyperbolic cosine and sine functions respectively. The standard notations are $\cosh x$ and $\sinh x$. Similarly one can define other hyperbolic trigonometric functions such as $\tanh x$ and $\coth x$.)

Solution. They are parallel to the case of the exponential function E. We only consider the uniqueness issue. As in the case for the exponential function, it suffices to show if both g satisfy $g'' = g$, $g(x_0) = g'(x_0) = 0$ at some x_0, then $g \equiv 0$. Well, it is a direct check that g satisfies the integral equation

$$g(x) = \int_{x_0}^{x} \int_{x_0}^{t} g(z)dz \ .$$

We claim $g \equiv 0$ on $[x_0 - 1, x_0 + 1]$. For, let $M = |g(x_1)|$ be the max of $|g|$ on this interval. We have

$$M = |g(x_1)| \leq \left| \int_{x_0}^{x} \int_{x_0}^{t} g(z)dz \right| \leq M \int_{x_0}^{x} \int_{x_0}^{t} dz = M \frac{(x - x_0)^2}{2} \leq \frac{M}{2} \ ,$$

which forces $M = 0$.

Remark. The functions c and s are actually the hyperbolic cosine and sine functions given respectively by

$$\cosh x = \frac{e^x + e^{-x}}{2} \ , \quad \sinh x = \frac{e^x - e^{-x}}{2} \ .$$